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ON THE THEORY OF FLEXURAL PIEZOCERAMIC CIRCULAR PLATE SOUND RADIATORS

S. Hanish

Transducer Branch
Sound Division

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U. S. NAVAL RESEARCH LABORATORY
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ABSTRACT

Circular electrostrictive plates, oppositely polarized and cemented together to form bilamellate disks, are submerged in a semi-infinite medium and driven by an applied electric field, or by applied acoustic pressures, to radiate sound. Formulas for plate velocity, acoustic power radiated, acoustic pressure, mechanical Q, etc., have been derived for the cases of centrally supported and edge supported disks in infinite baffles.

PROBLEM STATUS

This is an interim report on one phase of the problem; work on the problem is continuing.

AUTHORIZATION

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ON THE THEORY OF FLEXURAL PIEZOCERAMIC CIRCULAR PLATE SOUND RADIATORS

INTRODUCTION

A convenient source of underwater sound in the range of 100 cps to 50 kc is the family of flexural, circular, piezoceramic bilamellate disks. These, in diameters of 0.5 to 25 cm can radiate up to 1/2 to 1 watt per square centimeter for applied electric fields of 150 to 250 volts rms per millimeter of half thickness, pulsed power. A simplified theory of the performance characteristics of such sources is presented in this report. In the first part of the theory the piezoceramic disk is centrally supported, and in the second part the disk is simply supported on its edge. Both cases are treated only with reference to the presence of an infinite stiff baffle enclosing the half space into which the sources are radiating acoustic energy. The analysis proceeds on the assumptions of a Class I (Fischer's classification) piezoelectric transducer and not on the basic laws of the Class III transducer, to which the electrostrictive types belong. This has been done to avoid transduction ratios and lumped parameters which are frequency dependent. As a result of this choice, the material constants appearing in the course of the paper are only "effective" piezoelectric moduli, to be measured on particular samples of polarized ceramics by techniques conventionally applied to true piezoelectric crystals. In short, the ceramics have been converted to "effective" crystals, and standard piezoelectric theory has been applied to their behavior.

CENTRALLY SUPPORTED FLEXURAL DISKS

The Structure

The transducer (Fig. 1) is a bilamellar disk, one or both halves of which may be actively polarized electrostrictive ceramics. While it is always advantageous to have a disk of two active halves, the use of a central support may require half of the bilamellar structure to be a passive metal, e.g., brass. Each face of an active ceramic plate is completely electroded with fired silver paste. Permanent polarization of the ceramic is in the z direction, normal to the plate. The structure is driven into forced flexural vibration by the application of an alternating electric field E applied across the thickness of the ceramic plate.* Upon submersion, the disk radiates sound into a semi-infinite liquid medium of characteristic impedance $\rho_w c_w$, from a circular hole of equal diameter in an infinite stiff baffle.

Figure 2 shows the conventions of signs which hold in this analysis. The neutral axis of bending is set at $z = 0$. For small displacements w in the positive z direction, the displacement u in the x direction is $u = -z(\partial w/\partial x)$, where $\partial w/\partial x$ is positive, as shown. The strain S_{xx} in the x direction is $S_{xx} = \partial u/\partial x = -z(\partial^2 w/\partial x^2)$. We note that $\partial^2 w/\partial x^2$ is negative when w is positive and that the strain is positive (tension) for positive z. Similarly, for displacement v in the y direction, the strain in the y direction is $S_{yy} = -z(\partial v/\partial y) = -z(\partial^2 w/\partial y^2)$. A positive bending moment M_x or M_y is one which causes a deflection of the disk in the positive z direction, i.e., downward, as shown, and induces resisting stresses T_{xx} or T_{yy}.

*See List of Symbols at end of report.

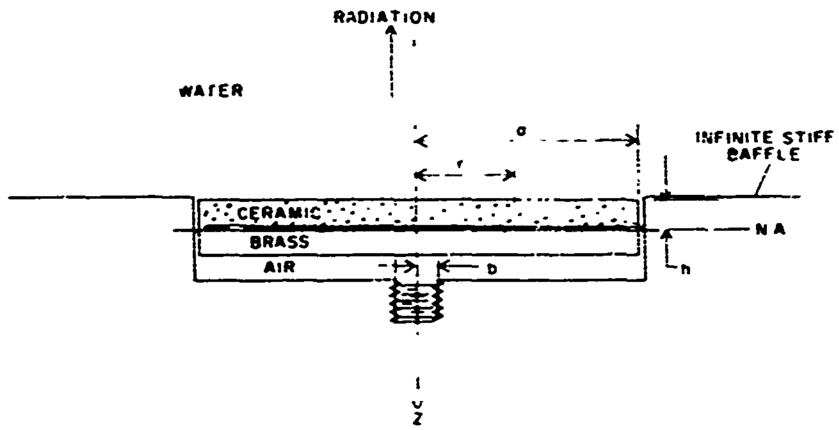

Figure 1 - Schematic cross section of a piezoceramic-brass circular plate sound radiator

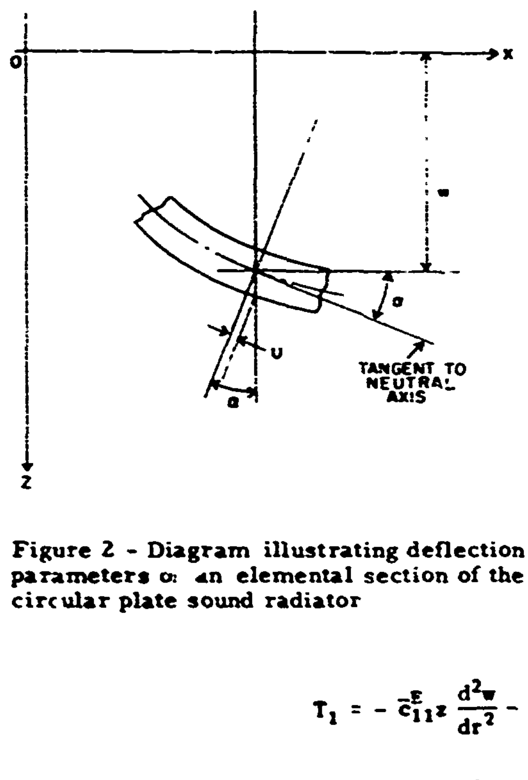

Figure 2 - Diagram illustrating deflection parameters of an elemental section of the circular plate sound radiator

The same conventions hold for a cylindrical coordinate system in which the origin corresponds to the origin of the xyz Cartesian system.

Equations of State

The circular bilamellar disk (half thickness h) is assumed "very thin," and the stress system correspondingly two-dimensional. This restriction to stresses in a plane is indicated at any stage in the analysis by a bar over all material constants (i.e., $\bar{c}_{rr}$, $\bar{c}_{r\theta}$, etc.). A cylindrical coordinate system is used, and, for convenience in writing, r, θ, z are designated by the numbers 1, 2, and 3. In such a planar cylindrical system the equations of state of the bilamellar disk under symmetrical electrical excitation may be immediately taken from the tensor form used by Mason (1).

$$T_1 = -\bar{c}_{11}^E z \frac{d^2 w}{dr^2} - \bar{c}_{12}^E z \frac{1}{r} \frac{dw}{dr} - \bar{e}_{31} E_3 . \tag{1a}$$

$$T_2 = -\bar{c}_{12}^E z \frac{d^2 w}{dr^2} - \bar{c}_{11}^E z \frac{1}{r} \frac{dw}{dr} - \bar{e}_{31} E_3 . \tag{1b}$$

$$D_3 = -\bar{e}_{31} z \left(\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr}\right) + \epsilon_{33}^S E_3 . \tag{1c}$$

The negative sign appears in the term $-\bar{e}_{31} E_3$ for the following reason. If a cylindrical pill of material is scooped from the interior of the ceramic, and if a positive gradient

E_3 is applied across the thickness, the pill is assumed to expand in the z direction, the polarization being correctly oriented for this expansion. Now if the material adjacent to the pill in the z direction is hindered from moving (condition of zero strain), it will develop a resisting compressive (negative) stress. Similarly, simultaneous contraction of the pill in the r and θ directions will induce tension stresses in the adjacent material which are the resisting stresses T_1 and T_2. Since the "effective" piezo modulus $\bar{e}_{31}$ is a negative number, a minus sign is added to the term $\bar{e}_{31} E_3$ so as to render T_1 and T_2 positive.

Corresponding to the internal resisting stresses T_1 and T_2 due to the applied field E_3 are the internal bending moments M_r, M_θ resulting from bilamellar action. By suitable integrations over the thickness of the plates, expressions are obtained for these bending moments in terms of displacement derivatives.

$$M_r = \int_{-h}^{+h} T_1 \cdot z\,dz = D\left(\frac{d^2 w}{dr^2} + \frac{\nu}{r}\,\frac{dw}{dr}\right) - \frac{\bar{e}_{31} h^2 E_3 \zeta}{2}\,. \tag{2a}$$

$$M_\theta = \int_{-h}^{+h} T_2\, z\,dz = D\left(\nu\,\frac{d^2 w}{dr^2} + \frac{1}{r}\,\frac{dw}{dr}\right) - \frac{\bar{e}_{31} h^2 E_3 \zeta}{2}\,. \tag{2b}$$

where

M_r is the internal bending moment per unit of circumferential length

M_θ is the internal bending moment per unit of radial length

D is the flexural constant, $= (2/3)\, h^3 \bar{c}_{11}^E$

ν is Poisson's ratio, which for planar stress is $\bar{c}_{12}^E / \bar{c}_{11}^E$

ζ is 1 or 2, depending, respectively, on whether the backing plate is inactive (i.e., is metal) or active (i.e., is a ceramic plate polarized for bilamellar action), assuming, if active, that active halves are connected in parallel.

Similarly, the average electric displacement in one plate is

$$D_3 = -\frac{\bar{e}_{31} h}{2}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\,\frac{dw}{dr}\right) + \epsilon_{33}^S E_3\,. \tag{3}$$

As the radiation of sound induces a real surface pressure on the face of the disk exposed to the medium, an expression is needed for the internal resisting shear Q_r per unit of circumferential length. Since the applied electric field E_3 is independent of the radial coordinate r, we may use conventional thin-plate theory and write (2)

$$Q_r = D\left(\frac{d^3 w}{dr^3} + \frac{1}{r}\,\frac{d^2 w}{dr^2} - \frac{1}{r^2}\,\frac{dw}{dr}\right). \tag{4}$$

Equations 1 to 4 are the foundation equations upon which the frame of the analysis is built. They carry with them the limitations that restrict the final formulas to narrow grounds of validity. In particular, they are piezoelectric in origin, linear, and planar, the electric fields are low in frequency and small in magnitude, the sound fields are due to infinitesimal displacements, an infinite stiff baffle is present, and edge effects, cement between plates, etc., are considered to affect the results in minor ways only.

Coefficient of Electromechanical Coupling

The mechanical and electrical equations (Eqs. 2 and 3) for the state of flexure must be satisfied together; that is, they arise from independent laws of nature and are not reducible to a single statement except by simultaneous solution. If the expressions for M_r and M_θ are added and the resultant solved for the displacement derivatives, we obtain

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} = \frac{M_r + M_\theta}{D(1+\nu)} + \frac{\bar{e}_{31}E_3\zeta}{D(1+\nu)},$$

i.e., mechanical curvature of the disk is expressed in terms of the internal bending moments representing the mechanical and electrical fields. Since the same quantity on the left appears in the electric field equation (Eq. 3), we have, upon substitution, a single equation combining the influence of both fields.

$$D_3 = -\frac{\bar{e}_{31}h}{2}\left(\frac{M_r + M_\theta}{D(1+\nu)}\right) + \epsilon_{33}^{s}E_3\left(1 + \frac{3}{8}\zeta\,\frac{2\bar{e}_{31}^{2}}{\bar{c}_{11}^{E}(1+\nu)\epsilon_{33}^{s}}\right). \qquad (5)$$

This equation is a statement that a charge per unit area on the electrodes of one plate accompanies the condition of flexure. The first term on the right-hand side is the mechanical or motional charge, and the second term is the charge due to applied field. The coupling effect of the mechanical field appears as an increment in the clamped dielectric constant ϵ_{33}^{s}, which takes the form of the second factor in the enclosed term. Such a group of symbols appears in planar piezoelectric analysis and is conveniently designated as the coefficient of electromechanical coupling in the flexure mode. Bechmann (3) defines the mixed planar coefficient of electromechanical coupling $(k_p^2)_{mix}$ by the relation

$$(k_p^2)_{mix} = \frac{2\bar{e}_{31}^{2}}{\bar{c}_{11}^{E}(1+\nu)\epsilon_{33}^{s}}.$$

For the state of flexure, therefore, the coefficient of electromechanical coupling k_f^2 is

$$k_f^2 = \frac{3}{8}\zeta(k_p^2)_{mix}. \qquad (6)$$

We note that this equation is independent of the manner in which the disk is mechanically supported. When the structure has one active half and is metal backed, $k_f^2 = 3/8\,(k_p^2)_{mix}$. Equation (6) reveals an important feature of flexural sound radiators, namely that the interconversion of electrical and mechanical energy is markedly reduced by flexure from the latent potential of the planar state to $3/8\ \zeta$ of this potential. This reduction of coupling is balanced, of course, by the increased compliance of mechanical structure and the consequent lowering of natural frequencies of vibration. In many applications, the latter advantage more than overrides the resulting poorer power-handling capacity of the structure.

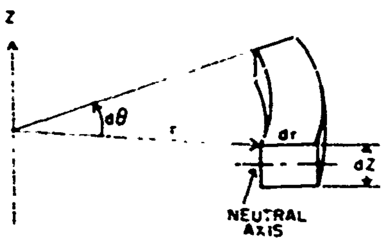

Figure 3 - Elemental volume of a circular plate sound radiator

Mechanical Displacement Under Forced Electrical Drive

The mechanical displacements of a piezoceramic disk which radiates sound into a semi-infinite liquid medium may be found by application of conventional elastic and acoustic theory. Figure 3 shows an elemental volume of the disk in cylindrical coordinates; m is the mass per unit of area and M_T the total differential bending moment per unit of circumferential length. Equating internal force per unit circumferential length on m (F_T in the z direction) to the acceleration of m, we have

$$F_T = -\frac{dM_T}{dr} = m(r\,d\theta\,dr)\,\frac{d^2w}{dt^2}.$$

Now to quantities of the first order in infinitesimals, we see from Fig. 3 that

$$M_T = \left[\left(M_r + \frac{dM_r}{dr}\,dr\right)(r + dr)\,d\theta - M_r(r\,d\theta) - 2\left(M_\theta\,dr\,\frac{d\theta}{2}\right) + Q_r\,r\,dr\,d\theta\right]$$

$$= \left(M_r + r\,\frac{dM_r}{dr} - M_\theta + r\,Q_r\right) dr\,d\theta .$$

Letting the primed symbol represent differentiation with respect to r, the force-acceleration equation becomes

$$2M_r' + rM_r'' - M_\theta' + rQ_r' + Q_r = -rm\,\frac{d^2w}{dt^2}.$$

Although M_r' and M_θ' are independent of the applied electric field, the magnitude of the elastic constants $\bar{c}_{11}$, $\bar{c}_{12}$, etc., depends upon this field. For low electric fields, we assume D_3 very small and use the constants $\bar{c}_{11}^D$, $\bar{c}_{12}^D$ instead of the constant $\bar{c}_{11}^E$, $\bar{c}_{12}^E$, etc., where by definition

$$\bar{c}_{11}^D = \bar{c}_{11}^E\,(1 + k_t^2)$$

$$\bar{c}_{11}^D = \bar{c}_{12}^E\,(1 + k_f^2).$$

For the state of larger electric fields, D_3 is hardly zero, and we must use a value of $\bar{c}_{11}$ that lies between $\bar{c}_{11}^E$ and $\bar{c}_{11}^D$. In all cases, however, k^2 is, at most, about 0.10, and the correct choice for the constants is seen to be no critical matter. Substituting for M_r', M_θ', thus modified for the particular magnitude of drive, we obtain

$$\frac{d^4w}{dr^4} + \frac{2}{r}\,\frac{d^3w}{dr^3} - \frac{1}{r^2}\,\frac{d^2w}{dr^2} + \frac{1}{r^3}\,\frac{dw}{dr} + \frac{m}{D^*}\,\frac{d^2w}{dt^2} = \frac{Q_r}{rD^*} + \frac{1}{D^*}\,\frac{dQ_r}{dr},$$

where

$$D^* = -\frac{2}{3}\,h^3\,\bar{c}_{11}^D .$$

The internal shear force Q_r per unit of circumferential length depends upon $q(r)$, the external load intensity per unit area. Static equilibrium requires that

$$2\pi r Q_r = \int_0^r 2\pi r\,dr\;q(r),$$

from which

$$\frac{Q_r}{rD^*} + \frac{1}{D^*}\,\frac{dQ_r}{dr} = \frac{q(r)}{D^*}.$$

From the assumptions of thin-plate theory, we restrict $q(r)$ to acoustic loads; i.e., $q(r)$ is the reaction acoustic pressure $p(r)$. In the condition of steady-state sinusoidal vibration, the equation of motion becomes

$$\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - k_1^2\right)\left(\frac{d}{dr^2} + \frac{1}{r}\frac{d}{dr} + k_1^2\right)\right] W = \frac{p(r,\omega)}{D^*}, \tag{7}$$

where

$$k_1^4 = -\frac{m\omega^2}{D^*} = \frac{3\rho_p\,\omega^2}{h^2\,\bar{c}_{11}^D}.$$

The surface pressure $p(r,\omega)$ may be determined from the following considerations. Since each element of area dA and velocity $\dot{W}$ radiates a spherical wave into the half space, the surface acoustic pressure is a summation of the pressure interaction effects of these waves. We have, from Fischer (4),

$$p(r,t) = j\,\frac{\rho_w c_w k}{2\pi}\int_A \dot{W}\,\frac{e^{-jkR}}{R}\,dA,$$

where R is the distance from an arbitrary point P in the plane of the disk to an element of area $RdRd\phi$ and ϕ is the angle between R and the line P drawn to the origin of coordinates. For the steady state, $\dot{W}$ is equal to $j\omega W$, and since $p(r,\omega)$ is a negative quantity, we obtain the final elastic-acoustic equation of motion

$$\nabla^4 W - k_1^4 W = \frac{\rho_w c_w^2 k^2}{2\pi D^*}\int_A W\,e^{-jkR}\,dR\,d\phi, \tag{8}$$

where $\nabla^4 W - k_1^4 W$ equals the left-hand side of Eq. 7.

Except for very simple geometries and for simple displacement distributions, this integro-differential equation is intractable. We can avoid this difficult mathematical situation by making the right-hand side independent of the variable r, which means in effect independent of the variable W. The reaction pressure, which is known to be a function of the coordinate r and the frequency ω, is replaced by an approximate pressure constant with radius but strongly dependent on the wavelength of radiation, that is, on frequency. Instead then of seeking an exact solution to the radiation problem posed by Eq. (8), we will seek the solution to a germane problem in which the external reaction pressure of the liquid medium is arbitrarily defined to be independent of r, while still retaining dependence on ω. In place of Eq. (7), therefore, we will set the equation of motion to be

$$\nabla^4 W - k_1^4 W = \frac{p(a,\omega)}{D^*}, \tag{9}$$

where a is the outside radius of the disk.

The reaction pressure $p(a,\omega)$ can be only an average estimate of the true pressure $p(r,\omega)$. A convenient procedure for determining $p(a,\omega)$ is to assume a displacement curve, $W(r)$ for the flexed disk, conformable to the boundary conditions, and perform the integration required by the right-hand side of Eq. (8). The result of this first step is the pressure function $p(r,\omega)$. McLachlan (5) carried through such an integration for the assumed distribution $\dot{W} = \dot{W}_0[1 - \tau(r^2/a^2)]$ and found that

$$p(r,t) = \rho_w c_w \dot{W}_0\left[\frac{z^2}{2!}g_2 - \frac{z^4}{4!}g_4 + \frac{z^6}{6!}g_6 - \cdots + j\left(zg_1 - \frac{z^3}{3!}g_3 + \frac{z^5}{5!}g_5 - \cdots\right)\right], \tag{10}$$

where g_2 and g_6 are known algebraic functions of the radius r, and g_1 and g_5 are known hypergeometric functions $F(\alpha,\beta,\gamma,\delta)$ of the same coordinate. The symbol z stands for the wave parameter ka. The infinite series of Eq. (10) is applicable to any size disk having the velocity distribution prescribed. In general, however, the frequencies corresponding to the grave mode of vibration are so low that ka is less than 1/2. We assume, then, that $ka \ll 1/2$ and write, after McLachlan (5),

$$p(r,t) = \rho_w c_w \dot{W}_o \left[\frac{(ka)^2}{4} + j(ka)g_1\right] \tag{11}$$

for the case of the centrally supported disk. The exact expression for g_1 is

$$g_1 = \frac{1}{3} F\left(-\frac{3}{2}, \frac{1}{2}, 1, b^2\right) + b^2 \left[F\left(-\frac{1}{2}, \frac{1}{2}, 1, b^2\right) - \frac{1}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 2, b^2\right)\right]$$

$$b = \frac{r}{a}.$$

The average pressure $p(a,t)$ is found by averaging Eq. (11) with respect to disk area. The result is

$$p(a,t) = \rho_w c_w \dot{W}_o \left[\frac{(ka)^2}{4} + j\,\frac{56}{45\pi}\,ka\right]. \tag{12}$$

A convenient symbol for the impedance portion of this equation is $Z(a,\omega)$, defined by the relation

$$p(a,t) = Z(a,\omega)\,\dot{W}_o$$

$$Z(a,\omega) = \rho_w c_w \left[\frac{(ka)^2}{4} + j\,\frac{56}{45\pi}\,ka\right].$$

Having now defined the nature of the expression $p(a,t)$, we return to Eq. (9) and substitute our results, recalling that in the steady state, $\dot{W} = j\omega W_o$. We have, then,

$$\nabla^4 W - k_1^4 W = \frac{j\omega W_o Z(a,\omega)}{D^*}. \tag{13}$$

This is an inhomogeneous differential equation, the right-hand side of which is independent of the coordinate r. A particular solution is easily seen to be

$$W = -\frac{j\omega W_o Z(a,\omega)}{D^* k_1^4}.$$

The complementary solution of the homogeneous equation is found in various texts, e.g., Ref. 6. Adding the particular solution to the complementary solution, we obtain the complete solution

$$W(r,\omega) = \alpha J_0(kr) + \beta Y_0(kr) + \gamma I_0(kr) + \delta K_0(kr) - \frac{j\omega Z(a,\omega) W_o}{D^* k_1^4}, \tag{14}$$

where α, β, γ, and δ are four constants, to be determined from the conditions of support.

The boundary conditions for the case of a flexural disk supported by a rigid built-in central pin (radius b) are easy to formulate, but they may not correspond to the actual stress configuration because of local yielding, rotary inertia effects, etc. At the risk then of stipulating what may be an ideal situation at the point of support, we specify that

at $r = b$	at $r = a$
1. $W = 0$	3. $M_r = 0$
2. $\frac{dW}{dr} = 0$	4. $Q_r = 0$.

Four conditions are thus available for determining the four constants α, β, γ, and δ by employing explicitly the general solution contained in Eq. (14). Upon performing the necessary derivations, we come to

$$\alpha J_0(k_1 b) + \beta Y_0(k_1 b) + \gamma I_0(k_1 b) + \delta K_0(k_1 b) = \frac{j\omega Z(a,\omega) W_o}{D^* k_1^4} \tag{15a}$$

$$-k[\alpha J_1(k_1 b) + \beta Y_1(k_1 b) - \gamma I_1(k_1 b) + \delta K_1(k_1 b)] = 0 \tag{15b}$$

$$\alpha A_1 + \beta A_2 + \gamma A_3 + \delta A_4 = -\frac{3}{4}\frac{\bar{e}_{31}}{\bar{c}^D_{11}}\frac{E_3}{k_1^2 h} \tag{15c}$$

$$k_1^3 [\alpha B_1 + \beta B_2 + \gamma B_3 + \delta B_4] = 0 \tag{15d}$$

where

$$A_1 = -J_0(k_1 a) + (1-\nu) J_1(k_1 a)/k_1 a$$

$$A_2 = -Y_0(k_1 a) + (1-\nu) Y_1(k_1 a)/k_1 a$$

$$A_3 = -I_0(k_1 a) - (1-\nu) I_1(k_1 a)/k_1 a$$

$$A_4 = K_0(k_1 a) + (1-\nu) K_1(k_1 a)/k_1 a$$

$$B_1 = J_1(k_1 a), \quad B_2 = Y_1(k_1 a), \quad B_3 = I_1(k_1 a), \quad B_4 = -K_1(k_1 a).$$

Simultaneous solution of Eqs. (15a, b, c, d) is rapidly performed by use of Cramer's rule. We can, however, simplify the results by noting that for centrally supported disks the ratio b/a is considerably less than unity, so much less, in point of construction, that we may assume that $b \to 0$ without major error. As a consequence of this choice, Eq. (15b) reduces to

$$\lim_{b \to 0} [\beta Y_1(k_1 b) + \delta K_1(k_1 b)] = 0,$$

or

$$\frac{2}{\pi}\beta - \delta = 0.$$

Substituting this result in Eq. (15a), and again letting $b \to 0$, we obtain

$$\alpha + \gamma = \frac{j\omega Z(a,\omega) W_o}{D^* k_1^4}.$$

Completing the solution for the two remaining conditions, Eqs. (15c) and (15d), and substituting the results in Eq. (14), we obtain

$$W(r,\omega) = \left\{ \frac{-\eta\left(B_2 + \frac{2}{\pi} B_4\right)}{\Delta} + \phi\, \frac{\left[B_3\left(A_2 + \frac{2}{\pi} A_4\right) - A_3\left(B_2 + \frac{2}{\pi} B_4\right)\right]}{\Delta} \right\} [J_0(k_1 r) - I_0(k_1 r)] \cdot$$

$$+ \left[\frac{\eta(B_1 - B_3)}{\Delta} + \phi\, \frac{A_3(B_1 - B_3) - B_3(A_1 - A_3)}{\Delta}\right] \left[Y_0(k_1 r) + \frac{2}{\pi} K_0(k_1 r)\right]$$

$$+ \phi[I_0(k_1 r) - 1] \cdot , \tag{16}$$

where

$$\eta = \frac{3}{4}\, \frac{\bar{e}_{31} E_3}{\bar{\epsilon}_{11}^{D}\, h k_1^2}, \qquad \phi = \frac{j\omega Z(a,\omega) W_o}{D^* k_1^4}$$

$$\Delta = (A_1 - A_3)\left(B_2 + \frac{2}{\pi} B_4\right) - (B_1 - B_3)\left(A_2 + \frac{2}{\pi} A_4\right).$$

The important quantity, of course, is the displacement at the outer edge, W_o. At $r = a$,

$$W_o = \frac{-\eta\left\{\left(B_2 + \frac{2}{\pi} B_4\right)[J_0(k_1 a) - I_0(k_1 a)] - (B_1 - B_3)\left[Y_0(k_1 a) + \frac{2}{\pi} K_0(k_1 a)\right]\right\}}{\Delta - \frac{j\omega Z(a,\omega)}{D^* k_1^4}\left\{B_3\left[A_2 + \frac{2}{\pi} A_4\right] - A_3\left[B_2 + \frac{2}{\pi} B_4\right]\right\}[J_0(k_1 a) - I_0(k_1 a)]}$$

$$- \frac{j\omega Z(a,\omega)}{D^* k_1^4}\left\{A_3(B_1 - B_3) - B_3(A_1 - A_3)\right\}\left[Y_0(k_1 a) + \frac{2}{\pi} K_0(k_1 a)\right]$$

$$- \frac{\Delta\, j\omega Z(a,\omega)}{D^* k_1^4}[I_0(k_1 a) - 1]. \tag{17}$$

The edge (maximum) displacement W_o is proportional to the electric gradient E_3 (that is, proportional to the constant η). In the absence of acoustic load (i.e., when $Z = 0$) the expression for Δ becomes zero for an infinite number of values of ka. The lowest of these (excluding $ka = 0$) corresponds to the grave (or umbrella) mode of vibration; the remaining values of ka correspond to modes of vibration consisting of a successively increasing number of nodal circles. With the accession of an acoustic load the denominator, for certain values of ka, reaches a minimum, but it may never be zero, since $Z(a,\omega)$ is a complex quantity. We note in particular that it is the resistive part of $Z(a,\omega)$ which contributes an imaginary term to the denominator and that a purely reactive load will not restrain the motion of the disk at mechanical resonance.

Transduction Ratio

The time-varying charge Q accumulating on the electrodes of one plate is obtained by integrating the expression for the dielectric displacement, Eq. (3), with respect to plate area. We have (assuming $b \to 0$)

$$Q = \int_A D_3\, dA = -\bar{e}_{31} h\pi \int_0^a \left(\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr}\right) r\,dr + \pi a^2 \epsilon_{33}^{s} E_3$$

or

$$Q = -\pi \bar{e}_{31} h a \left[\frac{dW}{dr}\right]_{r=a} + \pi a^2 \epsilon_{33}^{s} E_3. \tag{18}$$

The alternating current due to an applied voltage V_3 (across one plate) is therefore

$$
\begin{aligned}
i = j\omega\pi\bar{e}_{31}hk_1a \Bigg\{ & \frac{-\eta\left(B_2 + \frac{2}{\pi}B_4\right)}{\Delta}\left[J_1(k_1a) + I_1(k_1a)\right] \\
& + \frac{\eta(B_1 - B_3)}{\Delta}\left[Y_1(k_1a) + \frac{2}{\pi}K_1(ka)\right] \\
& + \frac{j\omega Z(a,\omega)W_o}{D^* k_1^4}\left[\frac{B_3\left(A_2 + \frac{2}{\pi}A_4\right) - A_3\left(B_2 + \frac{2}{\pi}B_4\right)}{\Delta}\right]\left[J_1(k_1a) + I_1(k_1a)\right] \\
& + \frac{j\omega Z(a,\omega)W_o}{D^* k_1^4}\left[\frac{A_3(B_1 - B_3) - B_3(A_1 - A_3)}{\Delta}\right]\left[Y_1(k_1a) + \frac{2}{\pi}K_1(k_1a)\right] \\
& - \frac{j\omega Z(a,\omega)W_o}{D^* k_1^4} I_1(k_1a)\Bigg\} + j\omega C^s V_3 . \qquad (19)
\end{aligned}
$$

While the electric current thus found is a useful parameter in exploring the accustic performance of a centrally supported disk radiator, successful application of four-terminal network theory requires some simpler form than that presented by Eq. (19). To obtain a more convenient expression for i, we note, from Eq. (16), that to a first approximation in kr, the deflection W has a parabolic distribution with respect to the coordinate r; that is, $W \sim r^2$. If then we assume a deflection curve of second order in r,

$$W = W_o(r^2/a^2),$$

we find

$$\left(\frac{dW}{dr}\right)_{r=a} = \frac{2}{a}W_o \qquad \text{(second-degree terms in } W \text{ only).}$$

The charge for a condition of parabolic distribution becomes

$$|Q| = N\,W_o + \pi a^2 \epsilon_{33}^s E_3 , \qquad (20)$$

where $N = 2\pi\bar{e}_{31}h = \pi\bar{e}_{31}t$.

The symbol N is the ratio of electric charge accumulating on the electrodes of one plate to the peak mechanical displacement (at the edge of the disk). To the approximation of second-degree terms in r for the variation of mechanical displacement with radius, N is seen to be a real number. When fourth-order terms (and higher) in the deflection curve are considered, the magnitude of the slope diminishes, and with it the transduction ratio N. To find the transduction ratio corresponding to higher order deflection curves, we assume more complex displacements, consonant with the boundary conditions, or, alternatively, we assume some known mechanical load on the surface of the disk. Two examples of the former method and one example of the latter are given below.

Example 1. Let the displacement curve be of the form

$$W = W_o\left[A\left(\frac{r}{a}\right)^2 + B\left(\frac{r}{a}\right)^4\right].$$

This type of variation with radius has been used by investigators in recent times. Southwell (7) has shown that such an assumed curve yields a value of grave resonant frequency 75 percent of the correct figure, if $A = 1$ and $B = -0.275$. Proceeding with these assigned values of A and B, we obtain

$$a\left(\frac{dW}{dr}\right)_{r=a} = 0.9$$

and

$$|N| = 0.45\pi\bar{e}_{31}t .$$

Example 2. Let the displacement curve be the one which Southwell has shown to yield a value of grave resonant frequency 99 percent of the true value; that is, let

$$W = W_o\left[r^q - \frac{q^2(q-2)}{4}a^{q-2}r^2\log\frac{r}{a}\right]$$

$$q = 2.89 .$$

Proceeding with the integration, we find that

$$|N| = 0.52\pi\bar{e}_{31}t .$$

Example 3. Let the disk be loaded on its surface with a uniform real static pressure p_o. It is easy to show that the total charge $|Q|$ from one plate is

$$|Q| = \frac{3}{4}\,\frac{p_o\pi a^4 d_{31}}{t^2} = \frac{3}{4}\,\frac{a^2}{t^2}d_{31}F ,$$

where $F = p_o\pi a^2$.

From the theory of elasticity,

$$W_o = \frac{Fa^2}{\pi Y_o^E t^3} .$$

Hence

$$|Q| = \frac{3}{4}\pi Y_o^E d_{31}tW_o$$

$$= 0.525\pi\bar{e}_{31}tW_o \qquad \text{(for } \nu = 0.3\text{)}.$$

Therefore $|N| \approx 0.53\,\pi\bar{e}_{31}t .$

The inclusion of terms to fourth order in the slope $(dW/dr)_{r=a}$ reduces the transduction ratio found for parabolic distribution by a factor of 2. Correspondingly, the presence of an external static pressure load introduces a similar reduction in N. Terms higher than fourth order affect these results in minor ways only. We conclude that the transduction ratio under actual operating conditions lies between $|N| = \pi\bar{e}_{31}t$ and $|N| = (\pi/2)\bar{e}_{31}t$.

We have now reached a point in the analysis where further progress is impeded by the complexity of the approximate deflection curve as revealed by Eqs. (16) and (17). It is more convenient, from this point on, to assume a deflection curve of simple algebraic form, consonant with the boundary conditions, such that computations of acoustic power, etc., are facilitated. Such a choice would leave one factor, namely the peak displacement W_o, indeterminate. However, the transduction ratio N relates W_o to the charge Q and therefore relates mechanical force F to applied voltage V_3. With the magnitude of N explicitly known, the peak displacement, velocity, etc., become electrical quantities whose magnitudes are then precisely determinable. In accordance with this procedure, then, we choose a deflection curve of simple parabolic form, namely $W = W_o(r/a)^2$, and proceed.

Radiated Acoustic Power, Mechanical Reactive Power, Kinetic Energy, Resonant Frequency, Acoustic Pressure and Mechanical Q of a Centrally Supported Disk Whose Deflection Curve is Parabolic (Second Order in r). Infinite Baffle Present.

In the far field, at a great distance R from the infinite baffle in which the disk with parabolic deflection is located, the radial distribution of real acoustic pressure p_a has been found by McLachlan (8) to be

$$p_a = \frac{\rho_w \ddot{W}_o a^2}{R}\left[\frac{J_1(z)}{z} - \frac{2\,J_2(z)}{z^2}\right] \tag{21}$$

$$z = ka \sin\theta \qquad k = \frac{2\pi}{\lambda} \qquad \lambda = \text{wavelength.}$$

Since the liquid-particle velocity v at great distances is $p_a/\rho_w c_w$, the peak sound power P_a radiated into semi-infinite space is

$$P_a = \int_A p_a v\, dA = 2\pi \frac{\rho_w}{c_w} a^4 \ddot{W}_o^2 \int_0^{\frac{\pi}{2}} \left[\frac{J_1(z)}{z} - \frac{2\,J_2(z)}{z^2}\right]^2 \sin\theta\, d\theta$$

or

$$P_a = 2\frac{\rho_w}{c_w} \dot{W}_o^2 a^4 \sum_{m=0}^{\infty} \frac{(-1)^m\, 2^m (ka)^{2m}}{(2m+1)\,(2m-1)\ldots}$$

$$\left[\frac{\left(\frac{1}{2}\right)^{2+2m}(2+2m)!}{(2+m)!\,(1+m)!} + \frac{4\left(\frac{1}{2}\right)^{4+2m}(4+2m)!}{(4+m)!\,(2+m)!^2} - \frac{4\left(\frac{1}{2}\right)^{3+2m}(3+2m)!}{(3+m)!\,(1+m)!\,(2+m)!}\right]. \tag{22a}$$

When this expression is expanded, the peak real power becomes

$$P_a = 2\pi \frac{\rho_w}{c_w} \omega^4 W_o^2 a^4 \left[\frac{1}{16} - \frac{(ka)^2}{72} + \frac{5}{3456}(ka)^4 - \cdots\right]. \tag{22b}$$

At the frequency of mechanical resonance in the grave (umbrella) mode, the magnitude of ka is usually less than 1/2. Limiting the infinite series of the above equation to the first term only, we obtain

$$P_a = \frac{\pi}{8}\frac{\rho_w}{c_w} \dot{W}_o^2 \omega^2 a^4 \qquad \text{(peak power for condition } ka \ll 1/2\text{).} \tag{22c}$$

The mechanical reactive power P_x can be derived by a similar procedure applied to the reactive pressure p_i, explicitly written in Eq. (10). As in the above case, we limit the expression for p_i to the first term in ka and write

$$P_x = \int_A p_i v\, dA$$

$$= j\rho_w c_w 2\pi a^2 \dot{W}_o^2 (ka) \int_0^1 g_1 b^3\, db\,.$$

Since the value of the integral is $20/63\pi$, we have

$$P_x = j\,\omega L_1 \dot{W}_o^2 \tag{23a}$$

$$L_1 = \frac{40}{63}\,\rho_w a^3 \,. \tag{23b}$$

The parameter L_1 is the inertial mass which the liquid medium adds to the disk flexing in the grave mode. It coincides with the values found by McLachian (5).

The magnitude of the kinetic energy of vibration (T_p) of the disk with no acoustic load is also found by a simple integration:

$$T_p = \int_{b\to 0}^{a} \frac{1}{2}\,2\pi r t\;\rho_p \dot{W}_o^2 \left(\frac{r}{a}\right)^4 dr$$

or

$$T_p = \frac{M}{6}\dot{W}_o^2 , \qquad M = \pi a^2 \rho_p t \,. \tag{24}$$

The lumped mass, corresponding to the edge velocity $\dot{W}_o$, is therefore 1/3 of the actual disk mass. Adding to this lumped mass the inertial mass of the water (L_1), we obtain the total kinetic mass M_q, corresponding to $\dot{W}_o$;

$$M_q = \frac{\pi a^2 t}{3}\,\rho_p(1+\beta) \tag{25a}$$

$$\beta = \frac{120}{63\pi}\;\frac{\rho_w a}{\rho_p t} \,. \tag{25b}$$

The presence of a semi-infinite liquid medium may be thought to raise the density of the plate from its value ρ_p to $\rho_p(1+\beta)$. In an ordinary design for a water medium $\rho_w a/\rho_p t$ is close to unity, making β about 2/3. One result of this added mass is to lower the natural resonant frequency ω_o of the disk in the grave mode from its value in a vacuum (approximated by air) by the factor $(1+\beta)^{-1/2}$. An expression for ω_o may be obtained by solving the secular equation

$$\Delta = 0 , \tag{26}$$

where Δ is the denominator of Eq. (16) in the absence of acoustic load. We obtain as the first root of this equation the value $ka = 1.933$, a magnitude quite close to $ka = 1.937$ found by Southwell (7). Upon solving for ω_o, the vacuum resonant frequency becomes

$$\omega_o = 1.081\,\frac{t}{a^2}\,c_p , \qquad c_p = \left(\frac{\bar{c}_{11}^{D}}{\rho_p}\right)^{1/2} . \tag{27}$$

Hence the resonant frequency in a liquid medium (ω_R) becomes

$$\omega_R = 1.081\,\frac{t}{a^2}\,c_p^* , \qquad c_p^* = \left(\frac{\bar{c}_{11}^{D}}{\rho_p(1+\beta)}\right)^{1/2} . \tag{28}$$

A spherical wave (such as is radiated by the disk for the condition $ka \ll 1/2$) of source strength Q_s, looking into a semi-infinite medium bounded by a stiff baffle, develops a pressure p in the far field (distance R) whose expression (4) is

$$p = j\,\frac{Q_s}{2\pi}\,\rho_w \omega\,\frac{e^{-jkR}}{R} . \tag{29a}$$

We omit the time-dependent factor. If the velocity distribution over the surface of the disk is parabolic,

$$Q_s = \int_A v_s \, dA = \int_0^a \dot{W}_o \left(\frac{r}{a}\right)^2 2\pi r \, dr$$

$$= (\pi a^2/2) \dot{r}_o .$$

From this, the absolute magnitude of far-field pressure is seen to be

$$|p| = \frac{\rho_w \omega^2 a^2}{4R} W_o . \quad (29b)$$

Another important performance parameter of the disk sound radiator is its natural damping factor α_d, or its steady-state mechanical Q_m. For assumed parabolic deflection, the rms radiated power (P_{rms}) is

$$P_{rms} = \frac{\pi}{16} \omega^2 \frac{\rho_w}{c_w} \dot{W}_o^2 a^4 .$$

The rms kinetic energy of the plate and water is one-half of the magnitude found previously, since this quantity varies as $\cos^2 \omega t$. In any cycle of vibration, however, the mean energy T_m is twice the kinetic energy. Hence the mean energy of plate and water is

$$T_m = \frac{\pi a^2 \rho_p (1+\beta) t}{6} \dot{W}_o^2 .$$

Now the diminution of mean energy during free vibration is equal to the radiated power; that is,

$$\frac{dT_m}{dt} = - P_{rms} ,$$

or

$$\frac{\pi \rho \, (1+\beta) t \, \dot{W}_o \ddot{W}_o a^2}{3} = - \frac{\pi}{16} \omega^2 \frac{\rho_w}{c_w} \dot{W}_o^2 a^4 .$$

From this,

$$\ddot{W}_o = - \alpha_d \dot{W}_o ,$$

where

$$\alpha_d = \frac{3}{16} \frac{\rho_w a^2 \omega^2}{\rho_p t \, c_w (1+\beta)} . \quad (30)$$

The quantity α_d is the temporal damping factor for free vibration, when the damping is due entirely to radiation. With it we can derive an expression for the number of cycles of free vibration (at resonant frequency ω_R) that must be completed for the amplitude $\dot{W}_o$ to be reduced to $1/e$ of its original maximum value. This is $\omega_R/2\pi\alpha_d$, and hence

$$\frac{\omega_R}{2\pi\alpha_d} = 0.788(1+\beta)^{3/2} \frac{\rho_p c_w}{\rho_w c_p} .$$

Now by definition, the mechanical Q is

$$Q_m = \frac{\omega_R}{2\alpha_d}.$$

Therefore

$$Q_m = 2.47(1+\beta)^{3/2}\ \frac{\rho_p c_w}{\rho_w c_p}. \tag{31}$$

Mechanical Displacement Under Forced Mechanical Drive and Receiving Response

A steady-state (sinusoidal) pressure of magnitude p_o (independent of the parameter ka) is allowed to drive the submerged, centrally supported disk. We stipulate that $ka \ll 1/2$ and write the equation of motion as

$$\nabla^4 W - k_1^4 W = \varphi + \varphi', \tag{32}$$

where

$$\varphi = \frac{j\omega Z(a,\omega)}{D^*}, \qquad \varphi' = \frac{p_o}{D^*}.$$

As for boundary conditions, we require that $D_3 = 0$ (i.e., open circuit). Assuming once again that $b/a \ll 1$, we obtain for the first two boundary conditions

$$\alpha + \gamma = \varphi + \varphi'$$

$$\frac{2}{\pi}\beta - \delta = 0.$$

Proceeding with the two remaining boundary conditions, solving for α, β, γ, and δ on the assumption that E_3 is zero, and substituting in the displacement equation, we obtain

$$W_o = \frac{\dfrac{p_o}{D^*}\left\{\left[A_3\left(B_2+\frac{2}{\pi}B_4\right) - B_3\left(A_2+\frac{2}{\pi}A_4\right)\right][J_0(k_1a)-I_0(k_1a)] + [B_3(A_1-A_3)-A_3(B_1-B_3)]\left[Y_0(k_1a)+\frac{2}{\pi}K_0(k_1a)\right]\right.}{\Delta - \dfrac{j\omega Z(a,\omega)}{D^* k_1^4}\left[B_3\left(A_2+\frac{2}{\pi}A_4\right) - A_3\left(B_2+\frac{2}{\pi}B_4\right)\right][J_0(k_1a)-I_0(k_1a)] - \dfrac{j\omega Z(a,\omega)}{D^* k_1^4}[A_3(B_1-B_3)-B_3(A_1-A_3)]\left[Y_0(k_1a)+\frac{2}{\pi}K_0(k_1a)\right] - \dfrac{j\omega Z(a,\omega)}{D^* k_1^4}[I_0(k_1a)-1]}. \tag{33}$$

The complexity of this expression requires that some simpler representation be sought for the accumulated charge $|Q|$ due to the pressure p_o than is contained in the requirement that $|Q| = \pi\bar{e}_{31} h a (dW/dr)_{r=a}$. We assume, as before, a parabolic deflection curve, and calculate the open-circuit voltage V_3 due to $|Q|$ stored upon a capacitor of area πa^2, thickness h, and dielectric constant ϵ_{33}. We have then

$$\frac{V}{p_o} = \frac{\bar{e}_{31}}{4}\ \frac{t^2\, n\, W_o}{a^2\, \epsilon_{33}^S\, p_o}, \tag{34}$$

where $n = 2$ for parabolic displacements. We can use the same formulation for the case of fourth-order terms by setting $n = 1$.

It is useful, at this point, to derive an expression for the receiving response at frequencies so low that $Z(a,\omega)$ becomes negligibly small, that is, at quasi-static load conditions. From the theory of elasticity,

$$\frac{W_o}{p_o} = \frac{a^4}{t^3 (Y_o^D)_{AV}} \qquad \text{(static load)},$$

where $(Y_o^D)_{AV}$ is the average Young's modulus at zero current.

Upon substitution, we obtain

$$\frac{V}{p_o} = \frac{1}{4}\,\frac{a^2}{t}\,\frac{\bar{e}_{31} n}{\epsilon_{33}^{s} (Y_o^D)_{AV}} \qquad \text{(static load).} \tag{35}$$

Formulas Describing the Acoustic Performance of a Metal-Ceramic Bilamellate at Mechanical Resonance, Radiating Sound into a Liquid Medium. Parabolic Deflection Curve Assumed and Infinite Baffle Present.

At mechanical resonance in the medium, the magnitude of edge velocity $\dot{W}_o$ is governed solely by the radiation resistance of the medium to the disk vibrator. Since the real acoustic pressure for the condition $ka \ll 1/2$ is independent of radius, the total mechanical force restraining motion at mechanical resonance is $F_R = p\pi a^2$. We know that edge velocity and pressure are related through the impedance equation, Eq. (10). Hence

$$F_R = \frac{\rho_w c_w (ka)^2 \pi a^2}{4} \dot{W}_o . \tag{36}$$

When electrically driven by a voltage V_3, the applied (mechanical) voltage force NV_3 balances the mechanical resistance force F_R in the steady state. Recalling that $N = \pi \bar{e}_{31} t$ for the condition of parabolic distribution, we find, for edge velocity,

$$\dot{W}_o = \frac{4NV}{\rho_w c_w (ka)^2 \pi a^2} ,$$

or

$$\dot{W}_o = 1.71\, \bar{e}_{31} \frac{c_w E_3}{c_p^{*2} \rho_w} . \tag{37}$$

This is a key equation, for with it, and with Eq. (28) for the resonant frequency ω_R, we can substitute into Eq. (22c) to obtain an explicit equation for the acoustic power

$$P_a = 5.36\, \bar{e}_{31}^{2} \frac{c_w V_3^2}{\rho_w c_p^{*2}} . \tag{38}$$

Similarly, substituting Eq. (37) into Eq. (29b), we find that the (real) acoustic pressure in the far field (distance R) for $ka \ll 1/2$ is

$$|p| = \frac{0.924\, \bar{e}_{31} c_w}{R\, c_p^{*}} V_3 . \tag{39}$$

As for receiving response (V/p_o), when an alternating pressure of constant magnitude p_o is applied (at the mechanical resonant frequency) to one side of the bilamellate, and when simultaneously the electric terminals are open-circuited, an edge velocity is initiated whose magnitude is

$$\dot{W}_o = \frac{4p_o}{\rho_w c_w (ka)^2}.$$

In consequence of the electromechanical coupling action, an alternating electric current $i = N\dot{W}_o$ flows through one plate of the disk and charges the capacitance C^s to a potential

$$V = \frac{Q}{C^s} = \frac{N\dot{W}_o}{j\,\omega C^s} = \frac{4\,p_o\,N}{j\,\rho_w c_w (ka)^2 \omega_R C^s}.$$

Substituting for ω_R and N, we obtain

$$\left(\frac{V}{p_o}\right)_{\omega=\omega_R} = 1.582\,\frac{a^2\,\bar{e}_{31}\,c_w}{t\;\epsilon_{33}^s\,\rho_w\,c_p^{*3}}. \tag{40}$$

The open-circuit voltage reported by this equation depends, for its magnitude, on the presence of an infinite stiff baffle. In the absence of such a baffle, the magnitude of response would fall by a factor of 2. The response, in any case, is that across one of the two possible active plates of the disk.

Conclusion

Equations (6), (27), (31), (37), (38), (39), and (40) are the most useful results of this part of the analysis. Taken together, they form a relatively complete summary of the acoustic performance of a centrally supported piezoceramic. One seemingly vital equation appears missing, namely an expression for the motional admittance. This is not a serious omission, since the small magnitude, of flexural electromechanical coupling renders the bilamellar almost a pure capacitor, when loaded by a liquid medium. In many instances the motional resistance is less than 1/10 of the reactance at mechanical resonance, and for moderate power absorption the current flow through the radiation resistance is relatively small. No great error therefore ensues in treating the bilamellate as a "pure" capacitor. As for material constants, it is best to determine these by actual test; e.g., it is best to determine c_p^* by measuring the resonant frequency in the liquid and computing this quantity from Eq. (28). Only in this way can one avoid dubious values for these parameters. Preliminary estimates of the values of these parameters based upon the available literature may be obtained from Table 1, which lists the piezoelectric properties of three popular piezoceramics.

An alternative procedure for determining the electrical series resistance R at velocity resonance is to assume that $R \approx X_B/Q_E$, where X_B is the blocked series reactance at resonance and Q_E is the electrical Q. The factor Q_E may be obtained from a knowledge of Q_m and k_f^2 by the additional approximation that $Q_E = 1 - k_f^2/k_f^2\,Q_m$. At velocity resonance therefore the electrical impedance is approximately

$$Z_{(\omega=\omega_R)} \approx \frac{k_f^2\,Q_m X_B}{1-k_f^2} - j\,X_B\,.$$

TABLE 1
Material Constants of Polarized Ceramics

Electromechanical Property (See List of Symbols)	Ceramic*		
	Ceramic B $BaTiO_3 - CaTiO_3$	PzT-4† $PbTiO_3 - PbzrO_3$	PzT-5† $PbTiO_3 - PbzrO_3$
$\bar{c}_{11}^E$ (planar stiffness)	127.9×10^9 (N/m^2)	91.3×10^9	74.7×10^9
$\bar{c}_{12}^E$ (planar stiffness)	38.3×10^9 (N/m^2)	27.9×10^9	23.0×10^9
s_{11}^E (compliance)	8.62×10^{-12} (m^2/N)	12.05×10^{-12}	12.05×10^{-12}
kp (planar coupling coefficient)	0.33	0.48	0.54
c_1 (strain-electric field modulus)	-58×10^{-12} (C/N)	-97×10^{-12}	-140×10^{-12}
$\bar{e}_{31}$ (planar piezo modulus)	-9.45 (N/mV)	-10.38	-12.01
ϵ_{33}^* (blocked dielectric constant)	9.64×10^{-9} (C/V$_m$)	7.89×10^{-9}	10.24×10^{-9}
ϵ_{33}^T (dielectric constant at zero stress)	10.7×10^{-9} (C/V$_m$)	9.7×10^{-9}	13.22×10^{-9}
ρ_c (ceramic density)	5.4×10^3 (kg/m^3)	7.6×10^3	7.6×10^3
ν (Poisson's Ratio)	0.3	0.3	0.3

*The chemical content of the above listed ceramics may be obtained by writing to the author of this paper.
†The two $PbTiO_3 - PbzrO_3$ mixes are different in composition.

EDGE-SUPPORTED BILAMELLATE DISKS

We shall consider in this part a bilamellate disk of two active halves, simply supported at its outer edge. Care must be taken, in building this structure, to insure simplicity of support, i.e., to insure absence of clamping while at the same time not restraining radial displacement along the outer rim. In any case, the boundary conditions $W = 0$, $M_r = 0$ must apply; that is, the external casing must restrain but not flex the disk.

Coefficient of Electromechanical Coupling, Bending Moments, Dynamic Equation, and Solutions

A few of the results of the previous derivations are immediately applicable, since the mode of mechanical support does not enter into the basic equations of state. From Eqs. (2a) and (2b), we obtain (since $\zeta = 2$),

$$M_r = D^* \left[\frac{d^2 W}{dr^2} + \frac{\nu}{r} \frac{dW}{dr} \right] - \bar{e}_{31} h^2 E_3 \tag{41a}$$

$$M_\theta = D^* \left[\nu \frac{d^2 W}{dr^2} + \frac{1}{r} \frac{dW}{dr} \right] - \bar{e}_{31} h^2 E_3 . \tag{41b}$$

Similarly, the coefficient of electromechanical coupling in the flexural (grave) mode is

$$k_f = \sqrt{\frac{3}{4}}\,(k_p)_{MIX}\,. \tag{42}$$

In order to write the dynamic equation, we need an expression for $p(a,\omega)$, as required by Eq. (9). Now the radial distribution of pressure $p(r,t)$ has a form identical with that expressed by Eq. (10), with the exception that the g's (hypergeometric functions) contain additional terms. Restricting ourselves once again to the condition $ka << 1/2$, we employ only factors g_1 and g_2 in the infinite series of Eq. (10). On the assumption that the deflection curve has the form $W = W_c[1-(r/a)^2]$, we consult McLachlan (5) and find that $g_2 = 1/2$, and g_1' is

$$g_1' = F\left(-\frac{1}{2}, \frac{1}{2}, 1, b^2\right) - \left\{\frac{1}{3}F\left(-\frac{2}{2}, \frac{1}{2}, 1, b^2\right) + b^2\left[F\left(-\frac{1}{2}, \frac{1}{2}, 1, b^2\right) - \frac{1}{2}F\left(-\frac{1}{2}, \frac{1}{2}, 2, b^2\right)\right]\right\}.$$

Performing once again the necessary integrations, we find that the reaction pressure, averaged over the surface area of the bilamellate disk, is

$$p(a,\omega) = j\,\omega W_c\, Z'(a,\omega) \tag{43a}$$

$$Z'(a,\omega) = \rho_w c_w\left[\frac{(ka)^2}{4} + j\,\frac{64}{45\pi}\,ka\right]. \tag{43b}$$

It may be concluded that a change in the mode of support has altered the pressure distribution in a minor way only, assuming, as has been done, that the deflection curve is parabolic in both conditions of support. Equations (43a) and (43b) lead directly, in conjunction with Eq. (13), to the equation of motion (steady state):

$$\nabla^4 W - k_1^4 W = \frac{j\,\omega W_c\, Z'(a,\omega)}{D^*}\,. \tag{44}$$

Although Eq. (14), with W_o replaced by W_c, is a general solution of the differential equation, the boundary conditions noted in the introduction to this section of the analysis require that β and δ be zero. Our general solution to Eq. (44) is therefore

$$W(r,\omega) = \alpha J_0(k_1 r) + \gamma I_0(k_1 r) - \frac{j\,\omega Z'(a,\omega)}{D^* k_1^4}\, W_c\,. \tag{45}$$

The two conditions that determine α, γ are

$$\alpha J_0(k_1 a) + \gamma I_0(k_1 a) = \frac{j\,\omega Z'(a,\omega) W_c}{D^* k_1^4}$$

$$\alpha A_1 + \gamma A_3 = -\frac{3}{2}\,\frac{\bar{e}_{31}\, E_3}{D_{11}\, k_1^2 h}\,.$$

Upon solving these simultaneously and substituting the results in Eq. (45), we obtain

$$W(r,\omega) = \frac{3}{2}\,\frac{\bar{e}_{31}\, E_3}{\bar{c}_{11}^D\, k_1^2 h}\left[\frac{I_0(k_1 a)\, J_0(k_1 r) - J_0(k_1 a)\, I_0(k_1 r)}{\Delta'}\right]$$

$$+ \frac{j\,\omega Z'(a,\omega)}{D^* k_1^4}\, W_c\left[\frac{A_3 J_0(k_1 r) - A_1 I_0(k_1 r)}{\Delta'}\right] - \frac{j\,\omega Z'(a,\omega)}{D^* k_1^4}\, W_c\,.$$

where
$$\Delta' = A_3 J_0(k_1 a) - A_1 I_0(k_1 a) . \tag{46}$$

From this, upon setting $r = 0$, we obtain

$$W_c = \frac{3}{2} \frac{\bar{e}_{31} E_3}{\bar{c}_{11}^D k_1^2 h} \frac{[I_0(k_1 a) - J_0(k_1 a)]}{\left[\Delta' + \frac{j\omega \Delta' Z'(a,\omega)}{D^* k_1^4} - \frac{j\omega Z'(a,\omega)(A_3 - A_1)}{D^* k_1^4}\right]} . \tag{47}$$

The displacement W, according to Eq. (46), is parabolic to a first approximation in the parameter kr. We have seen (Eq. 20) that for such a distribution of deflection, the transduction ratio N for one plate of the disk is $\pi \bar{e}_{31} t$. Under surface loads, however, the transduction ratio diminishes, as may be seen from the following development. Let the simply supported disk be subject to a static surface pressure p_o. The charge accumulating on one pair of plate electrodes is

$$Q = d_{31} \int_0^a \frac{s_r + s_t}{2} 2\pi r \, dr ,$$

where

d_{31} is the electric displacement-stress "effective" piezo modulus

s_r, s_t are the radial and tangential stresses induced in the disk by the load p_o.

From the theory of thin plates, it may be found that

$$\frac{s_r + s_t}{2} = -\frac{5}{2} p_o \frac{a^2}{t^2} \left[1 - \frac{4}{5} \frac{r^2}{a^2}\right]$$

and

$$p_o = -\frac{3}{2} \frac{Y_o t^3 W_c}{a^4} .$$

Upon performing the required integration, we obtain

$$Q = \frac{9}{8} \pi Y_o t \, d_{31} W_c$$

$$\approx \frac{3}{4} \pi \bar{e}_{31} t W_c .$$

that is, the transduction ratio N for the (static) loaded state is 3/4 of the value predicted by a parabolic deflection of the disk. When two plates are considered, the ratio, of course, will be twice the value of one plate. The advantage of using two active plates is thus clearly evident, since the coupling, power, pressure, etc., will all be improved.

Acoustic Power, Mechanical Reactive Power, Kinetic Energy, Mechanical Resonant Frequency and Mechanical Q of a Simply Supported Disk Whose Deflection Curve is Parabolic (Second Order in r). Infinite Stiff Baffle Present.

An expression for the acoustic pressure in the far field (distance R) due to a baffled disk having a deflection curve of the form $W = W_c[1 - (r/a)^2]$ is given by McLachlan (5):

$$p_a = \frac{2\rho_w a^2 \ddot{W}_c}{R} \frac{J_2(k_1 a \sin\theta)}{(k_1 a \sin\theta)^2}. \tag{48}$$

A suitable integration over the spherical area of radius R of the product of the acoustic pressure and the particle velocity yields the peak acoustic power radiated. In symbols,

$$P_a = \int_A p_a v dA = \frac{1}{\rho_w c_w} \int_A p^2 dA$$

$$= 8\pi \frac{\rho_w}{c_w} \ddot{W}_c^2 a^4 \int_0^{\frac{\pi}{2}} \frac{J_2^2(k_1 a \sin\theta)}{(k_1 a \sin\theta)^4} \sin\theta \, d\theta.$$

Upon evaluating the definite integral, we obtain

$$P_a = 8\pi \frac{\rho_w}{c_w} a^4 \ddot{W}_c^2 \sum_{m=0}^{\infty} \frac{(-1)^m (4+2m)!}{m!(4+m)!\ (2+m)!^2} \frac{\left(\frac{1}{2}\right)^{2(2+m)} (ka)^{2m} 2^m m!}{(2m+1)(2m-1)\ \ldots},$$

or

$$P_a = \pi \frac{\rho_w}{c_w} a^4 \ddot{W}_c^2 \left[\frac{1}{8} - \frac{(ka)^2}{72} + \cdots\right]. \tag{49}$$

We note that if $ka << 1/2$, the expression for peak acoustic power in the medium due to a flexible simply supported disk is the same as for a centrally supported disk, both having parabolic velocity distribution and both having equal maximum displacements.

To terms of first order in ka, the reactive pressure on the surface of the disk is

$$p_i = j \rho_w c_w \dot{W}_c (ka) g_i'.$$

The magnitude of reactive power in the medium (P_x) is therefore

$$P_x = j\omega 2\pi a^3 \rho_w \dot{W}_c^2 \int_0^1 g_i'(1-b^2)\, b\, db$$

or

$$P_x = j \frac{248}{315} \omega \rho_w a^3 \dot{W}_c^2. \tag{50}$$

The mechanical reactive power of the medium whose expression has just been derived is due entirely to the inertial effect of the medium. If the motion of each elementary volume of adjacent liquid is referred to the peak velocity $\dot{W}_c$, we see that the inertial mass added by the medium is $\rho_w a_x^3$ (248/315). Since the kinetic energy of the vibrating plate T_p is

$$T_p = \frac{1}{2} \int_0^1 \rho_w c_w \dot{W}_c^2 (ka)\, 2\pi a^2 (1-b^2)\, b\, db$$

$$= \frac{1}{6} \pi a^2 \rho_p t \dot{W}_c^2,$$

we conclude that the total effective mass of plate and water (M_q') referred to the peak velocity $\dot{W}_c$ is

$$M_q' = \frac{1}{3}\pi a^2 \rho_p t\,(1+\beta')$$

$$\beta' = \frac{3}{\pi}\left(\frac{248}{315}\right)\frac{\rho_w a}{\rho_p t} = 0.752\,\frac{\rho_w a}{\rho_p t}. \tag{51}$$

By thus increasing the effective mass of the plate, the liquid medium acts to reduce the natural frequency of vibration in air ω_o' by the factor $(1+\beta')^{-1/2}$. An expression for ω_o' is obtainable from Eq. (46) by setting $Z(a,\omega)$ equal to zero and solving the secular equation $\Delta' = 0$ for the lowest root, excluding $ka = 0$. We obtain, for this lowest root, a value $ka \approx 2.252$, from which we find that the grave resonant frequency of a liquid-loaded, simply supported plate to be

$$\omega_R' = 1.468\,\frac{t}{a^2}\,c_p'^{*} \tag{52}$$

$$c_p'^{*} = \left(\frac{\bar{c}_{11}^{D}}{\rho_p(1+\beta')}\right)^{1/2}.$$

Since for equal maximum displacements ($\dot{W}_o = \dot{W}_c$) the radiated acoustic power is the same whether the disk is simply supported or centrally supported, and since for parabolic displacements the kinetic energies of the unloaded plate are also identical, we see that the temporal damping factor α_d' has the same form as in Eq. (30), with the exception that β is replaced by β' and ω_R is replaced by ω_R'. The mechanical Q, therefore, becomes

$$Q_m = \frac{\omega_R'}{2\alpha_d} = 1.81\,(1+\beta')^{3/2}\,\frac{\rho_p c_w}{\rho_w c_p} \tag{53}$$

where

$$c_p = \frac{\bar{c}_{11}^{D}}{\rho_p}.$$

Formulas Describing the Acoustic Performance of a Simply Supported Ceramic Bilamellate, at Mechanical Resonance, Radiating Sound into a Liquid Medium. Parabolic Deflection Curve Assumed, and Infinite Baffle Present.

In this section the same reasoning and the same steps in derivation are applied as was used in obtaining Eqs. (36) to (40), with the additional consideration, however, that the transduction ratio N is, for the entire disk, 2X the value previously used. A first result, employing $N = 2\pi\bar{e}_{31}t$, is the center velocity, $\dot{W}_c$. We obtain, for this key parameter,

$$\dot{W}_c = 1.85\,\frac{\bar{e}_{31}\,c_w E_3}{c_p'^{*2}\rho_w}. \tag{54}$$

Similarly, using Eqs. (49) and (52) and this value for $\dot{W}_c$, we derive the expression for the acoustic power radiated

$$P_a = 11.65\,\bar{e}_{31}^{2}\,\frac{c_w V_3^2}{\rho_w c_p'^{*2}}. \tag{55}$$

As for the acoustic pressure, it is noted that when $z \ll 1/2$ the limit of $J_2(z)/z^2$ is 1/8. Hence the resonant acoustic pressure in the far field (distance R) on the acoustic axes has the same form as in Eq. (29b), with the exception that $\dot{W}_c$ replaces $\dot{W}_o$ and ω_R' replaces ω_R. Substituting as before for ω_R' and $\dot{W}_c$, we obtain

$$|p| = 1.36\,\bar{e}_{31}\,\frac{c_w V_3}{c'^{*} R}\,. \tag{56}$$

A procedure similar to that which led to Eq. (40) yields the resonant open-circuit receiving response

$$\left(\frac{V}{P_o}\right)_{\omega=\omega_R} = 0.632\,\frac{a^2}{t}\,\frac{\bar{e}_{31}\,c_w}{\epsilon_{33}^{s}\,\rho_w\,c_p'^{*3}}\,. \tag{57}$$

We observe here the same precaution in the application of Eq. (57) as was noted in the application of Eq. (40), namely that this is the response from a single plate (of two possible active plates) and that an infinite stiff baffle bounds the half space from which the incoming signal takes its origin. In the absence of the baffle, the response will fall to a value of one-half that noted above.

Conclusion

Equations (49), (52), (53), (54), (55), (56), and (57) constitute in their entirety a summary of the acoustic performance of a simply supported flexural bilamellate disk radiating sound into an infinite half space. All the limitations, precautions, etc., noted in the discussion at the conclusion of the earlier derivations are applicable to these last results. In particular, it is of importance to repeat the stipulation mentioned previously that all material constants occurring in the formulas be determined by test upon an actual disk. Substitutions from generalized data available in the literature may, or may not, lead to dependable results. And a final point: all power, pressure, and voltage response formulas are based on assumed 100-percent energy conversion, no losses occurring on the way. In actual practice, overall conversion efficiencies vary from 40 to 70 percent at low electric drive (0.01 volt rms per mil of thickness) to 15 to 20 percent at high, electric drive (10 volts rms per mil of thickness). In practice, too, the material constants $\bar{e}_{31}$ and $\bar{c}_{11}^{D}$ are lossy (i.e., are complex quantities) and are frequency sensitive. From these remarks, the approximate nature of the derived equations may be surmised.

LIST OF SYMBOLS

Symbol	Definition
A	area in square meter
a	radius of plate (meter)
C^s	capacitance (farads) at constant strain
c_{11}, c_{12}, etc.	stiffness moduli (N/m^2)
c_w	velocity of sound in water (m/sec)
D	stiffness constant, ($N \cdot m$)
D_3	electric displacement (coul/m^2)
e_{31}	piezo modulus
E_z, E_3	electric field (volt/m^1)
f	frequency (cps)
g_1, g_3, etc.	hypergeometric functions
h	half thickness of plate (m)
$J_n(z)$, $I_n(z)$, $Y_n(z)$, $K_n(z)$	Bessel Functions of order n and argument z
k	wave number, (m^{-1})
k_1	$3\rho_p\,\omega^2/h^2\,\bar{c}_{11}^D$
k_f	coefficient of electromechanical coupling in flexural mode
M_r	radial bending moment per unit of length ($N_{m/m}$)
M_θ	tangential bending moment per unit of length ($N_{m/m}$)
N	transduction ratio
P_a	acoustic power (watts)
p_a	real pressure
p_i	inertial reactive pressure
Q	charge (coulombs)
Q_m	mechanical Q

LIST OF SYMBOLS (Continued)

Symbol	Definition
Q_r	internal resisting shear per unit of circumferential length
R	distance in sound field (m)
r, θ, z	polar coordinates
S_1	radial strain (m/m)
S_2	tangential strain (m/m)
T_1	radial stress (N/m^2)
T_2	tangential stress (N/m^2)
V_3	applied voltage (volts)
v	velocity in meters per second
w	plate deflection in z direction as a function of time (m)
W	plate deflection in z direction as a function of frequency (m)
w, p	subscripts for water, plate
$Y_o E$	Young's modulus (N/m^2)
Z	acoustic impedance
z	ka
$\alpha, \beta, \gamma, \delta, \tau$	constants
ϵ_{33}^{s}	dielectric constant at constant (i.e., zero) strain (farad/m)
ζ	1 or 2, depending upon backing plate of bilamellate
ρ	density (kg/m^3)
ν	Poisson's ratio
ω	angular frequency
Δ	special denominator in Eq. (16)

<u>NOTE</u>: The MKS system is used throughout this paper.

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